The 2-spaces of the standard $E_6(q)$-module

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Let $\mathbb{F}$ be a field and let $\mathbb{K}$ be the vector space of dimension 27 over $\mathbb{F}$ whose elements are the triples $x = [x_1,x_2,x_3]$ with $x_i \in M_3(\mathbb{F})$, the set of 3×3-matrices with entries in $\mathbb{F}$, for $i = 1,2,3$, with addition and scalar multiplication taken entrywise. Set

$$\tilde{E} = \{g \in GL(\mathbb{K}): \text{ there is } \lambda \in \mathbb{F}^* \text{ such that, for all } x \in \mathbb{K}, \mathscr{D}(x^g) = \lambda\mathscr{D}(x)\},$$

where $\mathscr{D}:\mathbb{K}\to\mathbb{F}$ is the cubic form $\mathscr{D}(x) = \det x_1 + \det x_2 + \det x_3 - \operatorname{trace}(x_1x_2x_3)$. Then $\hat{E} = [\tilde{E},\tilde{E}]$ is the universal cover of the simple Chevalley group $E_6(\mathbb{F})$ (the split form) and $\tilde{E}$ is the extension of $\hat{E}$ by diagonal automorphisms. The center of $\hat{E}$ is $Z(\hat{E}) \cong \operatorname{Hom}(\mathbb{Z}_3,\mathbb{F}^*)$. Let e^i_{jk} be the element $[x_1,x_2,x_3]$ of $\mathbb{K}$ all of whose entries are 0 except for the j,k-entry of x_i, which is 1. Furthermore, set $e_i = e^1_{ii}$ for $i = 1,2,3$, and $e = e_1 + e_2 + e_3$. It is well known (cf. Mars [9]) that $\hat{E}$ has three orbits on P, the set of 1-spaces of $\mathbb{K}$, with representatives $<x>$, where x is one of e_1, $e_1 + e_2$, e. Clearly, $Z(\hat{E})$ fixes P pointwise, and the group $E = \hat{E}\,/\,Z(\hat{E})$ acts faithfully on P. Let $(\cdot,\cdot)$ be the inner product given by $(x,y) = \operatorname{trace}(x_1y_1 + x_2y_3 + x_3y_2)$ for $x = [x_1,x_2,x_3], y = [y_1,y_2,y_3] \in \mathbb{K}$. Define a map $\# :\mathbb{K}\to\mathbb{K}$, $x \mapsto x^\#$ by the identity

$$\mathscr{D}(x + ty) = \mathscr{D}(x) + (x^\#,y)t + (x,y^\#)t^2 + \mathscr{D}(y)t^3,$$

for $x,y \in \mathbb{K}$ and t an indeterminate over $\mathbb{F}$. (The map $\#$ can be explicitly defined as

$$x^\# = [x_1^\# - x_2x_3, x_3^\# - x_1x_2, x_2^\# - x_3x_1];$$

here $c^\#$, for a 3×3 matrix $c = (c_{ij})$, is the 3×3-matrix with i,j-entry $c^\#_{ij} = c_{kl}c_{mn} - c_{kn}c_{ml}$, where $(jkm) = (iln) = (123)$.) Thus $\#$ is a quadratic map and the set $\{<x> \in P : x^\# = 0\}$ is $\hat{E}$-invariant. The three $\hat{E}$-orbits of points $<x>$ in P can be distinguished by invariants as follows.

(i). $x^\# = 0$, $x \neq 0$. We refer to these $<x>$ as *white* points. For such a point, the stabilizer in $\hat{E}$ is $\hat{E}_{<x>} = QL$, a parabolic subgroup with unipotent radical $R_u(\hat{E}_{<x>}) = Q \cong \mathbb{F}^{16}$ and Levi complement L (so $Q \cap L = 1$) such that $[L,L] \cong \hat{D}_5(\mathbb{F})$, the spin group of type D_5 over $\mathbb{F}$ and $L/[L,L] \cong \mathbb{F}^*$.

(ii). $\mathscr{D}(x) = 0$, $x^\# \neq 0$. These will be called *gray* points. For such a point, $\hat{E}_{<x>} = QL$ where $Q = R_u(\hat{E}_{<x>}) \cong \mathbb{F}^{16}$, $Q \cap L = 1$, $[L,L] \cong \hat{B}_4(\mathbb{F})$, the spin group of type B_4 over $\mathbb{F}$, and $L/[L,L] \cong \mathbb{F}^*$.

(iii). $\mathscr{D}(x) \neq 0$. These we designate as *black* points. For such a point, $E_{<x>} \cong F_4(\mathbb{F})$.

The collections of white, gray, and black points will be designated by $\mathscr{W},\mathscr{G},\mathscr{B}$, respectively. Both $\mathscr{W}$ and $\mathscr{G}$ are $\hat{E}$-orbits. Two points $<x>,<y> \in \mathscr{B}$ are in the same $\hat{E}$-orbit if and only if $\mathscr{D}(x)\,/\,\mathscr{D}(y)$ has a cube root in $\mathbb{F}$.

We note that $<e> \in \mathscr{B}$ and that its stabilizer $F = \hat{E}_e \cong E_{<e>}$ preserves $(\cdot,\cdot)$, whence the map

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$x \mapsto x^{\#}$ $(x \in \mathbb{K})$. If $char(\mathbb{F}) \neq 3$, then $e \notin e^{\perp}$, where $\perp$ denotes the orthogonality relation induced in $\mathbb{K}$ by $(\cdot,\cdot)$, and $\mathbb{K} = \; <e> \oplus e^{\perp}$. In any case, we will let $V = \mathbb{K}/<e>$, a 26-dimensional module for F.

For $u,v \in \mathbb{K}$, we define $u \times v = (u+v)^{\#} - u^{\#} - v^{\#}$. This is an F-invariant symmetric, bilinear map from $\mathbb{K}$ to $\mathbb{K}$. If $U \subset \mathbb{K}$, we will also write $u \times U$ for the union of $u \times w$ over all $w \in U$. The group $\tilde{E}$ preserves the relation $x \times y = 0$ for $x,y \in \mathbb{K}$.

The present definition of the $\hat{E}_6(\mathbb{F})$-module, which is based on a Lie subgroup of type $A_2A_2A_2$ in the sense that such a subgroup obviously stabilizes $\mathcal{D}$ (see the section below), comes from FREUDENTHAL [6]. Other explicit forms of $\mathcal{D}$, related to subgroups of Lie type A_1A_5 and F_4, respectively, are given by DICKSON [5] (see also CHEVALLEY [2]) and JACOBSON [7]. More information of the module under study here can be found in recent work of ASCHBACHER [1].

It is our purpose to determine the orbits of $\tilde{E}$ on the 2-spaces (i.e., the 2-dimensional linear subspaces, also called the *projective lines*) of $\mathbb{K}$ under the assumption that $\mathbb{F} = \mathbb{F}_q$, a finite field with q elements. The main reason for restricting to finite fields is brevity of exposition; in fact it is not hard to extrapolate from the present text a qualitative result on the $\tilde{E}$-orbits of 2-spaces for the case of an arbitrary field.

Remark. When $\mathbb{F} = \mathbb{F}_q$ is finite of order q, the order of $\hat{E}$ is $q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)$ and the $\tilde{E}$-orbit sizes are as follows:

$$|\mathcal{W}| = \frac{(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)}, \quad |\mathcal{G}| = q^4\frac{(q^5-1)(q^{12}-1)(q^9-1)}{(q^4-1)(q-1)},$$

and $|\mathcal{B}| = q^{12}(q^9-1)(q^5-1)$.

The next five sections are organized as follows: Some useful elements of $\tilde{E}$ are explicitly given in Section A. In the subsequent section P we introduce some notation and preliminary facts that will prove useful throughout the analysis. In section W which then follows we will find all E-orbits of lines containing a white point. Then we move into section B determining the F-orbits of lines of $\mathbb{K}$ which contain $<e>$ and from this all $\tilde{E}$-orbits of lines which contain a black point. Finally, in section G we determine the E-orbits of lines containing gray points.

To end this section, we recall some standard notation for groups. We write Sym_n for the symmetric group on n letters and $\mathbb{Z}_n$ for the cyclic group of order n. Let p be a prime, and $q = p^a$ a power of p. We use q^n and $\mathbb{F}_q^n$ to denote the elementary abelian p-group $\mathbb{Z}_p^{an}$ and $[q^n]$ for a p-group of order q^n. The multiplicative group of the field $\mathbb{F}$ is denoted by $\mathbb{F}^*$ (or, if $\mathbb{F} = \mathbb{F}_q$, also by $\mathbb{Z}_{q-1}$). For groups A and B, $A.B$ denotes an extension with normal subgroup A and quotient B.

Section A. Some elements of $\tilde{E}$.

The following identity, first observed by FREUDENTHAL [6], has been used by SPRINGER [10] to characterize the pair of the bilinear form $(\cdot,\cdot)$ and $\mathcal{D}$.

$$x^{\#\#} = \mathcal{D}(x)x \quad (x \in \mathbb{K}).$$

From this identity, many others can be derived, cf. JACOBSON [8]. Here are three important ones, valid for all $x,y \in \mathbb{K}$:

$$x^{\#} \times (x \times y) = \mathcal{D}(x)y + (x^{\#},y)x,$$

$$x \times (x^{\#} \times y) = \mathcal{D}(x)y + (x,y)x^{\#},$$

$$(x \times y)^{\#} + x^{\#} \times y^{\#} = (x^{\#}, y)y + (x, y^{\#})x.$$

Using these, it is straightforward to derive that, for $x,y \in \mathbb{K}$ with $(x,y)=0$ and $<x>, <y> \in \mathscr{W}$, the map $t_{x,y}:\mathbb{K} \to \mathbb{K}$ given by

$$z^{t_{x,y}} = z + y \times (x \times z) - (z,y)x \quad (z \in \mathbb{K})$$

belongs to $\hat{E}$. It fixes x and the subspace $y \times \mathbb{K}$.

By H we denote the subgroup of $\hat{E}$ consisting of all elements which are diagonal with respect to the standard basis e^i_{jk} $(1 \leqslant i,j,k \leqslant 3)$. An arbitrary element $h \in H$ has the shape $h(\alpha,\beta,\gamma,\delta,\epsilon,\zeta) =$

$$\left[\begin{array}{ccc|ccc|ccc} \alpha\gamma^{-1}\delta^{-1} & \alpha\beta\gamma & \alpha\delta & \gamma\delta\zeta^{-1}\epsilon^{-1} & \beta\gamma\delta\epsilon & \gamma\delta\zeta & \alpha^{-1}\zeta\epsilon & \alpha\beta\zeta\epsilon & \epsilon\zeta \\ \alpha^{-1}\beta^{-1}\gamma^{-1}\delta^{-1} & \alpha^{-1}\gamma & \alpha^{-1}\beta^{-1}\delta & \beta^{-1}\gamma^{-1}\zeta^{-1}\epsilon^{-1} & \gamma^{-1}\epsilon & \beta^{-1}\gamma^{-1}\zeta & \alpha^{-1}\beta^{-1}\epsilon^{-1} & \alpha\epsilon^{-1} & \beta^{-1}\epsilon^{-1} \\ \gamma^{-1}\delta^{-1} & \beta\gamma & \delta & \delta^{-1}\zeta^{-1}\epsilon^{-1} & \beta\delta^{-1}\epsilon & \delta^{-1}\zeta & \alpha^{-1}\zeta^{-1} & \alpha\beta\zeta^{-1} & \zeta^{-1} \end{array}\right]$$

Here, each entry represents the scalar by which the corresponding basis element of $\mathbb{K}$ is multiplied in the action of h.

The diagonal transformations d_α $(\alpha \in \mathbb{F}^*)$ given by

$$\left[\begin{array}{ccc|ccc|ccc} \alpha^{-1} & 1 & 1 & \alpha & \alpha & \alpha & \alpha & 1 & 1 \\ 1 & \alpha & \alpha & 1 & 1 & 1 & \alpha & 1 & 1 \\ 1 & \alpha & \alpha & 1 & 1 & 1 & \alpha & 1 & 1 \end{array}\right]$$

with the same convention as for $h \in H$, complement H to a maximal diagonal subgroup of $\tilde{E}$ establishing that the morphism $\lambda:\tilde{E} \to \mathbb{F}^*$ given by $\mathscr{D}(x^g) = \lambda(g)\mathscr{D}(x)$ for $g \in \tilde{E}$ and $x \in \mathbb{K}$, is surjective.

The next set of transformations in $\hat{E}$ constitutes a subgroup isomorphic to a central product $SL_3(\mathbb{F}) \circ SL_3(\mathbb{F}) \circ SL_3(\mathbb{F})$. For $g_1,g_2,g_3 \in SL_3(\mathbb{F})$, we let s_{g_1,g_2,g_3} be the transformation s given by

$$x^s = (g_1 x_1 g_2^{-1}, g_2 x_2 g_3^{-1}, g_3 x_3 g_1^{-1}) \quad (x = [x_1,x_2,x_3] \in \mathbb{K}).$$

Many 2-spaces that we shall encounter have a conjugate in the 9-space $< e^1_{ij} : 1 \leqslant j,k \leqslant 3 >$; the subgroup $\{s_{g_1,g_2,id} : g_1,g_2 \in SL_3(\mathbb{F})\} \cong SL_3(\mathbb{F}) \times SL_3(\mathbb{F})$ stabilizes this 9-space and yields automorphisms suitable for deriving some of the transitivity results needed in the sequel.

Section P. NOTATION AND PRELIMINARY FACTS.

Let $\mathscr{I}$ be a collection of subspaces of $\mathbb{K}$ and M a subspace of $\mathbb{K}$. We will set

$$\mathscr{I}(M) = \{U \in \mathscr{I} : U \subseteq M\} \text{ and } M_{\mathscr{I}} = \{U \in \mathscr{I} : U \supseteq M\}.$$

When either $\mathscr{I}(M)$ or $M_{\mathscr{I}}$ consists of a single element we will, by abuse of notation, identify $\mathscr{I}(M)$ or $M_{\mathscr{I}}$ with this single element.

Next, suppose $\mathscr{I} \subseteq P$, that is, a set of 1-spaces of $\mathbb{K}$, and k is a positive integer. By $\mathscr{I}_k$ we will mean the collection of k-dimensional subspaces of $\mathbb{K}$ all of whose 1-spaces belong to $\mathscr{I}$. Thus, $\mathscr{I}_k = \{M \leqslant \mathbb{K} : \dim M = k, P(M) \subseteq \mathscr{I}\}$. A member of $\mathscr{W}_k$ will be called a *purely white k-space*, and a member of $\mathscr{G}_k$ a *purely gray k-space*. If $\mathbb{F} = \mathbb{F}_q$ and $l \in P_2$, we write $dist(l) = (|\mathscr{W}(l)|, |\mathscr{G}(l)|, |\mathscr{B}(l)|)$ for the *distribution* of the points of l among the $\tilde{E}$-orbits.

The permutation representation of E on $\mathscr{W}$ is a parabolic representation; it is well known and supports the Lie incidence structure $E_{6,1}(\mathbb{F})$ (cf. COHEN & COOPERSTEIN [3] and COOPERSTEIN [4]). The next several results can be deduced easily from facts about this representation.

(P.1). *The permutation rank of E on $\mathcal{W}$ is 3. For $w \in \mathcal{W}$, the orbits of $\hat{E}_w$ on $\mathcal{W}$ are $\{w\}$, $\Delta(w) = \{x \in \mathcal{W}: <w,x> \in \mathcal{W}_2\} = \{x \in \mathcal{W}: w \times x = 0\}$ and $\Gamma(w) = \{x \in \mathcal{W}: w \times x \neq 0\} = \{x \in \mathcal{W}: \mathcal{W}(<x,w>) = \{w,x\}\}$.*

We will write w' for the set $\{w\} \cup \Delta(w)$, and, if A is a subset of $\mathcal{W}$, we write A' for $\cap_{w \in A} w'$. We will also write x' instead of $<x>'$ if $<x> \in \mathcal{W}$. When $\mathbf{F} = \mathbf{F}_q$ we have

$|\Delta(w)| = q(q^4+1)(q^3+1)(q^2+1)(q+1)$ and $|\Gamma(w)| = q^8(q^5-1)(q^4+1)/(q-1)$.

(P.2). *Let $x,w \in \mathcal{W}$ with $x \in \Gamma(w)$. Set $S(x,w) = <w,x,x' \cap w'>$. Then $S(x,w) = (w \times x) \times \mathbf{K}$ and $\dim S(x,w) = 10$. For any $x_1, w_1 \in \mathcal{W}(S(x,w))$ with $x_1 \in \Gamma(w_1)$, we have $S(x_1,w_1) = S(x,w)$. Additionally, writing $S = S(x,w)$, we have*

(i) *$(\mathcal{W}(S), \mathcal{W}_2(S), \subseteq)$ is a polar space of type D_5 over $\mathbf{F}$;*

(ii) *the stabilizer $\hat{E}_S$ of S in $\hat{E}$ is a parabolic subgroup isomorphic, but not conjugate in $\hat{E}$, to $\hat{E}_w$;*

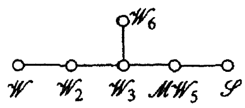
(iii) *$[R_u(\hat{E}_S), S] = 0$;*

(iv) *a Levi complement to $R_u(\hat{E}_S)$ in $\hat{E}_S$ is transitive on $P(S) \setminus \mathcal{W}(S)$, which consists of gray points.*

We will refer to the subspaces $S(x,w)$ as *symplecta* and denote the collection of such spaces by $\mathcal{S}$.

(P.3). *$\mathcal{W}_k \neq \emptyset$ if and only if $k \leqslant 6$. The stabilizer $\hat{E}_U$ of $U \in \mathcal{W}_k$ is a parabolic subgroup. If $k \neq 5$, E is transitive on $\mathcal{W}_k$. The group has two orbits on $\mathcal{W}_5$ which can be distinguished by maximality: one orbit consists of those purely white subspaces which are maximal under inclusion, the other consists of purely white subspaces which can be embedded in an element of $\mathcal{W}_6$.*

We will denote by $\mathcal{MW}_5$ the set of all elements of $\mathcal{W}_5$ which are maximal among the subspaces in $\mathcal{W}$. The Tits building associated with E can be obtained as a geometry whose elements of the various types are as indicated in the E_6-diagram below. Incidence is symmetrized inclusion except for unordered pairs from $\mathcal{MW}_5 \times \mathcal{W}_6$ and $\mathcal{S} \times \mathcal{W}_6$, (up to ordering), in which cases it is "meeting in a member of $\mathcal{W}_4$" and "$\mathcal{W}_5$", respectively.



(P.4). *If $S \in \mathcal{S}$, then S contains representatives of both E-orbits in $\mathcal{W}_5$.*

(P.5). *The permutation rank of E on $\mathcal{S}$ is 3. For $S \in \mathcal{S}$ the orbits of E_S on $\mathcal{S}$ are $\{S\}$, $\delta(S) = \{R \in \mathcal{S}: S \cap R \in \mathcal{MW}_5\}$, and $\gamma(S) = \{R \in \mathcal{S}: S \cap R \in \mathcal{W}\}$.*

We remark that the incidence system $(\mathcal{S}, \{U_{\mathcal{S}}: U \in \mathcal{MW}_5\}, \in)$ is isomorphic to $(\mathcal{W}, \mathcal{W}_2, \subseteq)$, the automorphism being established by an outer automorphism of E.

(P.6). *E has three orbits on $\mathcal{W} \times \mathcal{S}$. For any $w \in \mathcal{W}$ and $S \in \mathcal{S}$, precisely one of the following occurs:*

(i) *$w \in P(S)$;*

(ii) *$w' \cap S \in \mathcal{W}_5 \setminus \mathcal{MW}_5$;*

(iii) *$w' \cap S = 0$.*

When $\mathbf{F} = \mathbf{F}_q$, then, for $w \in \mathcal{W}$ and $S \in \mathcal{S}$, we have $|\{R \in \mathcal{S}: w \in P(R)\}| = |\mathcal{W}(S)| = (q^5-1)(q^4+1)/(q-1)$, $|\{R \in \mathcal{S}: w' \cap R \in \mathcal{W}_5\}| = |\{<x> \in \mathcal{W}: x' \cap S \in \mathcal{W}_5\}| = q^5(q^4+1)(q^3+1)(q^2+1)(q+1)$, and $|\{R \in \mathcal{S}: w' \cap R = 0\}| =$

$|\{x \in \mathcal{W} : x' \cap S = 0\}| = q^{16}$.

(P.7). *If $y \in \mathcal{G}$, then $|y_{\mathcal{S}}| = 1$, and $y_{\mathcal{S}} = y^{\#} \times \mathbb{K}$.*

Proof. Since E is transitive on $\mathcal{G}$ and $\mathcal{G}(S) \neq \varnothing$ for each $S \in \mathcal{S}$, we have $y_{\mathcal{S}} \neq \varnothing$. On the other hand, if $S \neq R \in \mathcal{S}$, then, by (P.5), $P(R \cap S) \subseteq \mathcal{W}$. It follows that $|y_{\mathcal{S}}| \leqslant 1$, and so $|y_{\mathcal{S}}| = 1$. The remaining assertion follows by a direct check for $y = e_1 + e_2$.

Section W. Lines with white points.

From now on we shall assume that $\mathbb{F} = \mathbb{F}_q$, a finite field of order q. The first result is a direct consequence of (P.2) and (P.3).

(W.1). *Let $l \in P_2$ and assume $|\mathcal{W}(l)| \geqslant 2$. Then either*

(i) *$l \in \mathcal{W}_2$ and $\hat{E}_l \cong [q^{25}].(SL_2(\mathbb{F}) \times SL_5(\mathbb{F})).\mathbb{F}^*$; or*

(ii) *$|\mathcal{W}(l)| = 2$, $P(l) \setminus \mathcal{W}(l) \subseteq \mathcal{G}$, $|l_{\mathcal{S}}| = 1$, and $\hat{E}_l \cong [q^{16}].\hat{D}_4(\mathbb{F}).(\mathbb{F}^*)^2.2$.*

We set $\mathcal{L}_1 = \mathcal{W}_2$ and $\mathcal{L}_2 = \{l \in P_2 : |\mathcal{W}(l)| = 2\}$. Then $|\mathcal{L}_i| = |\hat{E}| / |\hat{E}_l|$, for $l \in \mathcal{L}_i$, is as given in Table 1 below ($i = 1,2$).

(W.2). *Let $l \in P_2$, $w \in \mathcal{W}(l)$, $y \in \mathcal{G}(l)$ and assume $y_{\mathcal{S}} \cap w' = 0$. Then $P(l) \setminus \{w,y\} \subseteq \mathcal{B}$ and $\hat{E}_l = \hat{E}_{w,y} \cong \hat{B}_4(\mathbb{F}).\mathbb{F}^*$. There is a single $\tilde{E}$-orbit of such lines l. The $q-1$ black points of l occur in $(3,q-1)$ $\hat{E}_l$-orbits of length $(q-1)/(3,q-1)$.*

We denote the collection of such lines by $\mathcal{L}_3$.

Proof. By (P.6), E is transitive on pairs $(w,S) \in \mathcal{W} \times \mathcal{S}$ for which $w' \cap S = 0$. The subgroup $\hat{E}_{S,w} \cong \hat{D}_5(\mathbb{F}).\mathbb{F}^*$ is a Levi complement to the unipotent radical in each of the parabolic subgroups $\hat{E}_S$ and $\hat{E}_w$. It has has a single orbit on $\mathcal{G}(S)$ and, if $y \in \mathcal{G}(S)$, then $\hat{E}_{<w,y>} = \hat{E}_{w,y} = \hat{E}_{S,w,y} \cong \hat{B}_4(\mathbb{F}).\mathbb{F}^*$. It follows that $\hat{E}$ is transitive on $\mathcal{L}_3$. Take $w = \langle e_1 \rangle$, $y = \langle e_2 + e_3 \rangle$ and $l = \langle w,y \rangle$. Then l is a representative of such a line, and $P(l) \setminus \{w,y\} \subseteq \mathcal{B}$; a direct check, using elements of H, shows that elements $\langle u \rangle$ and $\langle v \rangle$ of $\mathcal{B}(l)$ are $\hat{E}_l$-conjugate if $\mathcal{D}(u)/\mathcal{D}(v)$ is a cube in $\mathbb{F}^*$. This completes the proof of (W.2).

(W.3). *The group F has two orbits in $\mathcal{W}$, with representatives $\langle e_1 \rangle$ and $\langle x \rangle$, where $x = e_{12}^1$. The respective stabilizers are $F_{<e_1>} \cong \hat{B}_4(\mathbb{F})$ and $F_{<x>} = QL$ a parabolic subgroup of F with unipotent radical $Q \cong \mathbb{F}^{15}$, and Levi complement isomorphic to $\hat{B}_3(\mathbb{F}).\ \mathbb{F}^*$.*

Proof. By (W.2), we have that $\hat{E}_{<e,e_1>} \cong \hat{B}_4(\mathbb{F}).\mathbb{F}^*$. From its action on $\mathcal{B}(\langle e,e_1 \rangle)$, it follows that $F_{<e_1>} \cong \hat{B}_4(\mathbb{F})$. This accounts for one orbit with $q^8(q^8 + q^4 + 1)$ points of $\mathcal{W}$. The geometry $(\mathcal{W}(e^{\perp}), \mathcal{W}_2(e^{\perp}), \subseteq)$ is the Lie incidence structure $F_{4,1}(\mathbb{F})$. From this it follows that $\mathcal{W}(e^{\perp})$ is an orbit of F containing $\langle x \rangle$, that $F_{<x>}$ is as stated, and $|\langle x \rangle^F| = (q^{12}-1)(q^8-1)/(q^4-1)(q-1)$. But then $|\langle e_1 \rangle^F| + |\langle x \rangle^F| = |\mathcal{W}|$, and hence all points in $\mathcal{W}$ have been accounted for.

(W.4). *The group $\tilde{E}$ has two orbits of lines l for which $\mathcal{W}(l) \neq \varnothing \neq \mathcal{B}(l)$. One orbit is $\mathcal{L}_3$. Denote the other orbit by $\mathcal{L}_4$. Then, for $l \in \mathcal{L}_4$, $|\mathcal{W}(l)| = 1$, $P(l) \setminus \mathcal{W}(l) \subseteq \mathcal{B}$, and $\hat{E}_l \cong [q^{16}].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$.*

Proof. Let $l \in P_2$ with $\mathcal{W}(l) \neq \varnothing \neq \mathcal{B}(l)$. Since $\tilde{E}$ is transitive on $\mathcal{B}$, without loss of generality we may assume $\langle e \rangle \in P(l)$. By (W.3), $F = \tilde{E}_e$ has two orbits on $\mathcal{W}$. Consequently, $\tilde{E}$ has two orbits on lines as described above, one of which must be $\mathcal{L}_3$. Suppose $l \in \mathcal{L}_4$. By (W.1), $|\mathcal{W}(l)| = 1$ and if $\{x\} = \mathcal{W}(l)$, then $P(l) \setminus \{x\} \subseteq \mathcal{B}$. Since F is transitive on $\mathcal{W}(e^{\perp})$, the group $\tilde{E}_x$ is transitive on $\{y \in \mathcal{B} : x \in \mathcal{W}(y^{\perp})\}$, from which it follows that $\tilde{E}_l = \tilde{E}_{x,l}$ is transitive on $\mathcal{B}(l)$. By (W.3), for

$y \in \mathscr{B}(l)$, we have $\hat{E}_{x,y} = [q^{15}].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$. Taking $x = <e_2^1>$ and $y = <e>$, elements of H suffice to establish transitivity on $l \setminus \{x,y\}$, whence transitivity of $\hat{E}_l$ on $\mathscr{B}(l)$. Thus, $\hat{E}_l \cong [q^{16}].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$ as claimed.

Remark. $\mathscr{L}_4$ consists of $(3,q-1)$ $\hat{E}$-orbits of equal length; they can be distinguished by the value (modulo cubes) in $\mathbb{F}^*$ of $\mathscr{D}$ on a black point of $l \in \mathscr{L}_4$.

(W.5). *Let $l \in P_2$. Assume $|\mathscr{W}(l)| = \{w\}$ for some $w \in \mathscr{W}$ and $l \setminus \mathscr{W}(l) \subseteq \mathscr{G}$. Then one of the following holds*

(i) $l_{\mathscr{S}} \neq \varnothing$, $w \times l = 0$, *and* $\hat{E}_l \cong [q^{24}].\hat{B}_3(\mathbb{F}).(\mathbb{F}^*)^2$;

(ii) $l_{\mathscr{S}} = \varnothing$, *for each* $y \in P(l) \setminus \{w\}$, *the space* $y_{\mathscr{S}} \cap w'$ *belongs to* $\mathscr{W}_5$, *and* $\hat{E}_l \cong [q^{22}].SL_4(\mathbb{F}).(\mathbb{F}^*)^2$.

In each case there is a single orbit, which we denote by $\mathscr{L}_5$ and $\mathscr{L}_6$, respectively.

Proof. Let $l \in P_2$ be such that $\mathscr{W}(l) = \{w\}$, and $P(l) \setminus \{w\} \subseteq \mathscr{G}$.
If $l_{\mathscr{S}} \neq \varnothing$, then, by (P.7), $|l_{\mathscr{S}}| = 1$. Put $S = l_{\mathscr{S}}$ and write $\hat{E}_S = QL$ where $Q = R_u(\hat{E}_S)$ and L a Levi complement. Then $[Q,S] = 0$, and L is transitive on $\{m \in P_2(S) : |\mathscr{W}(m)| = 1\}$. For such a line m, we have $L_m \cong [q^8].\hat{B}_3(\mathbb{F}).(\mathbb{F}^*)^2$. From this it follows that $\hat{E}_l = \hat{E}_{S,l} = QL_l$ is as stated above. Taking $w = <e_1>$ and $l = <e_2^1 + e_{21}^1, w>$, it is readily verified that $w \times l = 0$, whence (i).
Suppose now that $l_{\mathscr{S}} = \varnothing$. Let $y \in P(l) \setminus \{w\}$. Thus $y \in \mathscr{G}$. Set $S = y_{\mathscr{S}}$. If $w \in S$, then $l = <w,y> \subseteq S$ contradicting $l_{\mathscr{S}} = \varnothing$. If $w' \cap S = 0$, then, by (W.2), $P(l) \setminus \{w,y\} \subseteq \mathscr{B}$, again a contradiction since $P(l) \setminus \{w\} \subseteq \mathscr{G}$. According to (P.6), for all $y \in P(l) \setminus \{w\}$, it follows that $y_{\mathscr{S}} \cap w' \in \mathscr{W}_5$. Also by (P.6), $\hat{E}$ is transitive on pairs $(S_1,w_1) \in \mathscr{S} \times \mathscr{W}$ such that $S_1 \cap <w_1>' \in \mathscr{W}_5$. Let $\hat{E}_S = QL$ with $Q = R_u(\hat{E}_S)$ and L a Levi complement. $L_w = L_U$ where $U = S \cap w'$. Now $L_U \cong [q^{10}].SL_5(\mathbb{F}).(\mathbb{F}^*)^2$ is transitive on $\mathscr{G}(S)$ and, for $y \in \mathscr{G}(S)$, we have $L_{U,y} \cong [q^{6+4}].SL_4(\mathbb{F}).(\mathbb{F}^*)^2$. This shows that E is transitive on $\{(w,y) \in \mathscr{W} \times \mathscr{G}: w' \cap (y)_{\mathscr{S}} \in \mathscr{W}_5\}$, from which it follows that E is transitive on $\mathscr{L}_6 = \{l \in P_2 : |\mathscr{W}(l)| = 1, P(l) \setminus \mathscr{W}(l) \subseteq \mathscr{G}, l_{\mathscr{S}} = \varnothing\}$ and, for $l \in \mathscr{L}_6$, the subgroup $\hat{E}_l$ is transitive on $\mathscr{G}(l)$. The rest of (W.5) follows easily from this.

Now if $l \in P_2, \mathscr{W}(l) \neq \varnothing$, then l must satisfy one and only one of $|\mathscr{W}(l)| \geq 2$; $|\mathscr{W}(l)| = 1$ and $\mathscr{B}(l) \neq \varnothing$; or $|\mathscr{W}(l)| = 1$ and $P(l) \setminus \mathscr{W}(l) \subseteq \mathscr{G}$. We have therefore completely enumerated the lines l containing white points. We summarize:

(W.6). *If l is a 2-space of* $\mathbb{K}$ *containing a white point, then it is contained in $\mathscr{L}_i$ for a unique $i \in \{1,2,3,4,5,6\}$; furthermore, its stabilizer $\hat{E}_l$ and distribution $dist(l)$ are as given in Table 1.*

Table 1.
$\hat{E}$-orbits of lines l containing white points.

orbit	$\hat{E}_l$	orbit size	description	$dist(l)$
$\mathscr{L}_1$	$[q^{25}].SL_2(\mathbb{F}) \times SL_5(\mathbb{F}).\mathbb{F}^*$	$\frac{(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)}{(q^4-1)(q^3-1)(q^2-1)(q-1)}$	member of $\mathscr{W}_2$	$(q+1,0,0)$
$\mathscr{L}_2$	$[q^{16}].\hat{D}_4(\mathbb{F}).(\mathbb{F}^*)^2.2$	$\frac{q^8(q^{12}-1)(q^9-1)(q^5-1)(q^4+1)}{2(q^4-1)(q-1)^2}$	secant in symplecton	$(2,q-1,0)$
$\mathscr{L}_3$	$\hat{B}_4(\mathbb{F}).\mathbb{F}^*$	$\frac{q^{20}(q^{12}-1)(q^9-1)(q^5-1)}{(q^4-1)(q-1)}$	$<e_1,e_2+e_3>$	$(1,1,q-1)$

orbit	$\hat{E}_l$	orbit size	description	*dist*(l)
$\mathscr{L}_4$	$[q^{16}].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$	$q^{11}\frac{(q^{12}-1)(q^9-1)(q^8-1)(q^5-1)}{(q^4-1)(q-1)}$	$<e, e_{12}>$	$(1,0,q)$
$\mathscr{L}_5$	$[q^{24}].\hat{B}_3(\mathbb{F}).(\mathbb{F}^*)^2$	$q^3\frac{(q^{12}-1)(q^9-1)(q^5-1)(q^4+1)}{(q-1)^2}$	tangent in symplecton	$(1,q,0)$
$\mathscr{L}_6$	$[q^{22}].SL_4(\mathbb{F}).(\mathbb{F}^*)^2$	$q^8\frac{(q^{12}-1)(q^9-1)(q^8-1)(q^3+1)(q^5-1)}{(q^4-1)(q-1)^2}$	$<e_{21}, e_2+e_3>$	$(1,q,0)$

Section B. LINES WITH BLACK POINTS.

In this section $F = \hat{E}_e \cong F_4(\mathbb{F})$, $V = \mathbb{K}/<e>$. We will determine the orbits of F on V. From this we deduce the orbits of $\hat{E}$ on lines containing black points and the stabilizers of representatives. The main result is

(B.1). *The orbit structure of F on $P(V) = \{<e,x>/<e> : x \in \mathbb{K} \setminus <e>\}$ is given in Table 2, where ϵ is one of 2,3,4,5,7 and such that $q \equiv \epsilon$ (mod 6).*

Table 2.
F-orbits on $P(V)$.

orbit type	number of orbits	stabilizer $F_{<e,x>}$	orbit size	description of x	x in $e^\perp$
I	1	$\hat{B}_4(\mathbb{F})$	$q^8(q^8+q^4+1)$	$<x>$, $x^\# = 0$, $(x,e) \neq 0$	no
II	1	$[q^{15}].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$	$\frac{(q^{12}-1)(q^4+1)}{q-1}$	$<x>$, $x^\# = 0$, $(x,e) = 0$	yes
III	1	$[q^{14}].G_2(\mathbb{F}).\mathbb{F}^*$	$q^4\frac{(q^{12}-1)(q^8-1)}{q-1}$	$<x> \in \mathscr{G}$, $(x,e)=0$	yes
IV	1	$[q^7].\hat{B}_3(\mathbb{F})$	$q^8(q^{12}-1)(q^4+1)$	$<x> \in \mathscr{G}$, $(x,e) \neq 0$	no
V	$\frac{q-\epsilon}{6}$	$\hat{D}_4(\mathbb{F})$	$\frac{q^{12}(q^{12}-1)(q^8-1)}{(q^4-1)^2}$	generic in a special plane	no
VI	$\delta_{\epsilon,5}+\delta_{\epsilon,7}$	$\hat{D}_4(\mathbb{F}).2$	$\frac{q^{12}(q^{12}-1)(q^8-1)}{2(q^4-1)^2}$		no
VII	$\delta_{\epsilon,4}+\delta_{\epsilon,7}$	$\hat{D}_4(\mathbb{F}).3$	$\frac{q^{12}.(q^{12}-1)(q^8-1)}{3(q^4-1)^2}$		no
VIII	$\delta_{\epsilon,3}$	$\hat{D}_4(\mathbb{F}).\mathrm{Sym}_3$	$\frac{q^{12}(q^{12}-1)(q^8-1)}{6(q^4-1)^2}$		yes

orbit type	number of orbits	stabilizer $F_{<e,x>}$	orbit size	description of x	x in $e^{\perp}$
IX	$\frac{q+1-2\delta_{\epsilon,7}-2\delta_{\epsilon,4}-\delta_{\epsilon,3}}{3}$	${}^3D_4(\mathbb{F})$	$q^{12}(q^8-1)(q^4-1)$	generic in a triality twisted special plane	no
X	$2\delta_{\epsilon,7}+2\delta_{\epsilon,4}+\delta_{\epsilon,3}$	${}^3D_4(\mathbb{F}).3$	$\frac{q^{12}(q^8-1)(q^4-1)}{3}$		yes
XI	$\frac{q-1+\delta_{\epsilon,2}+\delta_{\epsilon,4}}{2}$	${}^2\hat{D}_4(\mathbb{F}).2$	$q^{12}(q^{12}-1)$	generic in a duality twisted generic plane	no
XII	$\delta_{\epsilon,7}+\delta_{\epsilon,5}+\delta_{\epsilon,3}$	${}^2\hat{D}_4(\mathbb{F}).2$	$\frac{q^{12}(q^{12}-1)}{2}$		yes

The stabilizers $[q^{14}].G_2(\mathbb{F}).\mathbb{F}^*$ in III and $[q^7].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$ in IV are contained in the stabilizer $[q^{15}].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$ in II. The stabilizers $\hat{D}_4(\mathbb{F}).2$, ${}^2\hat{D}_4(\mathbb{F})$, and ${}^2\hat{D}_4(\mathbb{F}).2$ in V, VI, XI, and XII, respectively, are contained in a stabilizer $\hat{B}_4(\mathbb{F})$ in I. If $char(F) = 3$, then V has a 25-dimensional irreducible submodule, namely $e^{\perp}/<e>$, whose orbit structure can be determined by the last column.

We will prove the result by establishing the existence of each of the orbits and showing they have the required stabilizer. Since the sum of the sizes of the orbits in the table is $(q^{26}-1)/(q-1)$, the assertion (B.1) will then follow. After this has been established, we will determine, for representatives l, the distribution $dist(l)$ and the action of E_l on $P(l)$.

(B.2). *The point $<x> = <e_1>$ is a representative of orbit I.*

Proof. Clearly $e_1^{\#} = 0$ and $(e,e_1) \neq 0$. In (W.3) we saw that $F_{<e,e_1>} = F_{<e_1>} \cong \hat{B}_4(\mathbb{F})$.

(B.3). *Let $x = e_{12}^1$. Then $x^{\#} = 0$, $(x,e) = 0$ and $F_{<e,x>} = F_{<x>} \cong q^{15}.B_3(\mathbb{F}).\mathbb{F}^*$ are as in II.*

Proof. This was shown in (W.3).

(B.4). *Let $x = e_{21}^1 + e_{13}^1$. Then $<x> \in \mathcal{G}$, $(x,e) = 0$ and $F_{<e,x>} = F_{<x>} \cong [q^{14}].G_2(\mathbb{F}).\mathbb{F}^*$ as in III.*

Proof. Let $y = e_{23}^1$. Then $x^{\#} = y$ and $(x,e) = 0$ as can be easily computed from the defining formulae for $(\cdot,\cdot)$ and $\#$. Note that $y^{\#} = 0$ and $(y,e) = 0$ and hence $<y>$ is a representative of the orbit in II. Set $Q = R_u(F_{<y>})$ and let L be a Levi complement to Q in $F_{<y>}$. It is easily computed that the symplecton $S = x_{\mathcal{S}} = y \times \mathbb{K} = <e_1, e_{21}^1, e_{13}^1, e_{23}^1, e_{21}^2, e_{22}^2, e_{23}^2, e_{13}^3, e_{23}^3, e_{33}^3>$ satisfies $y \in S$ and $S \cap e^{\perp} = <S \cap y'>$. The subgroup Q fixes every line on $<y>$ in $e^{\perp}$, and, for every such line m ($\neq <y,e>$ in case $char(\mathbb{F})=3$), is transitive on $P(m)\setminus\{<y>\}$. Thus $Q_{<x>}$ has index q in Q. Since $L \cong \hat{B}_3(\mathbb{F}).\mathbb{F}^*$ acts irreducibly on the 8-space $<S \cap y'>/<y>$ (as can be deduced from the F_4- geometry), it follows that L is transitive on the set of all lines $<y,z>$ of $<S \cap y'> = S \cap e^{\perp}$ for which $z^{\#} \neq 0$. Since $<y,x>$ is such a line, $L_{<y,z>} \cong G_2(\mathbb{F}).\mathbb{F}^*$. Hence $F_{<y,x>} \cong [q^{14}].G_2(\mathbb{F}).\mathbb{F}^*$. As $x^{\#} = y$, we conclude that $F_{<x>} = F_{<y,x>} \cong [q^{14}].G_2(\mathbb{F}).\mathbb{F}^*$ is as asserted.

(B.5). *Let* $x = e_{11}^{1} - e_{23}^{1}$. *Then* $<x> \in \mathcal{G}$, $(x,e) \neq 0$, *and* $F_{<x>} \cong [q^7].\hat{B}_3(\mathbb{F})$ *as in IV.*

Proof. Let y,S,Q,L be as in (B.4). Then $x^{\#} = y$ and $\mathcal{Q}(x)=0$, so $<x> \in \mathcal{G}$. Again, $x_{\mathcal{S}} = S$ and $F_{<x>} \subseteq F_{<y>}$. In this case, however, $x \notin S \cap e^{\perp} = <S \cap y'>$. The Levi complement L is transitive on the set of all lines m in S lying on $<y>$ for which $|\mathcal{W}(m)| = 2$. For any such line m, the unipotent radical Q is transitive the $q-1$ points of $\mathcal{G}(m)$. From this it follows that $F_{<x>} \cong [q^7].\hat{B}_3(\mathbb{F})$.

In the next several proofs we establish that the remaining $q^{25}-q^{20}-q^{16}$ points of V all belong to a so-called (twisted or untwisted) special plane. By definition (cf. ASCHBACHER [1]) a 3-space π in $\mathbb{K}$ is *special* if it has precisely 3 noncollinear white points and every point of π not on a line through two white points of π is black. There is a unique $\hat{E}$-orbit of such 3-spaces.

Let $\pi = <e_1,e_2,e_3>$. This is a special plane. Let $l_i = <e_j,e_k>$ where $\{i,j,k\} = \{1,2,3\}$. If $x = <u> \in P(\pi)$, then $\mathcal{Q}(u) \neq 0$ unless $x \in P(l_i)$ for some $i \in \{1,2,3\}$, and $u^{\#} \neq 0$ unless $x \in \{e_1,e_2,e_3\}$. Thus, $N_F(\pi)/C_F(\pi)$ is isomorphic to a subgroup of Sym_3. If $\{i,j,k\} = \{1,2,3\}$, we set $S_i = <e_j,e_k>_{\mathcal{S}}$. Then $S_i \cap <e_i>' = 0$. Since $\hat{E}_{S_i,e_i} \cong \hat{D}_5(\mathbb{F}).\mathbb{F}^*$, there exists an element $\tau_i \in \hat{E}_{S_i,e_i}$ interchanging e_j and e_k, and hence $\tau_i \in F = \hat{E}_e$. Then the subgroup $T = <\tau_1,\tau_2,\tau_3>$ of F fixes π and induces Sym_3 on $\{e_1,e_2,e_3\}$. Clearly $\hat{E}_{e_1,e_2,e_3} \cong \hat{D}_4(\mathbb{F})$ acts in each of its three distinct linear representation of degree eight on the spaces $\{e_1,e_2,e_3\}^{\perp} \cap (e_i \times \mathbb{K})$ where $i = 1,2,3$.

The following observation is crucial: if $x = <u> \in \pi \setminus \cup_{i=1}^{3} <e,e_i>$, then $\pi = <e,x,u^{\#}>$. It follows that π is the only F-conjugate of π containing $x = <u>$, and

$$\hat{D}_4(\mathbb{F}) \cong C_F(\pi) \leqslant F_x \leqslant F_{<e,x>} \leqslant N_F(\pi) \cong \hat{D}_4(\mathbb{F}).\mathrm{Sym}_3.$$

Hence $F_{<e,x>}/C_F(\pi)$ is a subgroup of Sym_3, completely determined by $|(<e,x>^F \cap P(\pi))/<e>|$. Let $x = <u> \in \pi \setminus \cup_{i=1}^{3} <e,e_i>$. As we are only interested in the F-orbit of $<e,u>$ modulo $<e>$ we may assume $u = e_2 + \alpha e_3$ for some $\alpha \in \mathbb{F} \setminus \{0,1\}$. Now the elements of T which induce (23), (12), (123), (132), (13) map $<e,u>/<e>$ onto $<e,e_2+\beta e_3>/<e>$ with $\beta = \alpha^{-1}$, $1-\alpha$, $1-\alpha^{-1}$, $(1-\alpha)^{-1}$, and $(1-\alpha^{-1})^{-1}$, in the respective cases. This implies that, for generic α, i.e., $\alpha \neq 0,1,-1,\frac{1}{2},2,\frac{1}{2}(1 \pm \sqrt{-3})$, we have $|(<e,x>^F \cap P(\pi))/<e>| = 6$. To be more specific in the remaining cases, that is, for non-generic α, we must consider the different values of q modulo 6.

(i). $q \equiv 1 \pmod 6$. The above 7 values for α are all distinct, so we get $(q-7)/6$ orbits with $F_x \cong \hat{D}_4(\mathbb{F})$. For $\alpha = -1,\frac{1}{2},2$ we have $|(<e,x>^F \cap P(\pi))/<e>| = 3$, and, for $\alpha = (1 \pm \sqrt{-3})/2$, we have $|(<e,x>^F \cap \pi)/<e>| = 2$. Thus, we get two additional orbits, one with $F_{<e,x>} \cong \hat{D}_4(\mathbb{F}).2$, the other with $F_{<e,x>} \cong \hat{D}_4(\mathbb{F}).3$.

(ii). $q \equiv 2 \pmod 6$. Thus q is an odd power of 2, and so $(1 \pm \sqrt{-3})/2$ (that is, primitive cube roots of 1) do not exist in $\mathbb{F}$. Since $2=0, 1=-1$, and $\alpha = \frac{1}{2}$ corresponds to $u = e_3$, we only need exclude $\alpha = 0,1$; thus there are $(q-2)/6$ orbits with stabilizer $F_{<e,x>} \cong \hat{D}_4(\mathbb{F})$.

(iii). $q \equiv 3 \pmod 6$. Now q is a power of 3, so $-1 = \frac{1}{2} = (1 \pm \sqrt{-3})/2$; with such an α we get an orbit with stabilizer $F_{<e,u>} \cong \hat{D}_4(\mathbb{F}).\mathrm{Sym}_3$. Excluding $\alpha = 0,1,-1$, we get $(q-3)/6$ orbits with $F_{<e,x>} \cong \hat{D}_4(\mathbb{F})$.

(iv). $q \equiv 4 \pmod 6$. Now q is an even power of 2, and two distinct primitive cubic roots of unity exist, leading to a single orbit with $F_{<e,x>} \cong \hat{D}_4(\mathbb{F}).3$. Excluding $\alpha = 0,1,(-1 \pm \sqrt{-3})/2$, there remain $(q-4)/6$ orbits with $F_{<e,x>} \cong \hat{D}_4(\mathbb{F})$.

(v). $q \equiv 5 \pmod 6$. Now $(1 \pm \sqrt{-3})/2$ do not exist, so we get (for $\alpha = -1,\frac{1}{2},2$) a simple orbit with $F_{<e,x>} \cong \hat{D}_4(\mathbb{F}).2$ and, for α generic, $(q-5)/6$ orbits with $F_{<e,x>} \cong \hat{D}_4(\mathbb{F})$.

Note that the lines $<e,u>$, for $u = e_2 + \alpha e_3$, contain three gray points and that the remaining $q-2$ points are black since they meet each of $< e_i,\hat{e}_j >$ in a point, but do not contain any of $<e_1>$, $<e_2>$, $<e_3>$. If $F_{<e,x>} \cong \hat{D}_4(\mathbb{F})$ or $\hat{D}_4(\mathbb{F}).2$ then at least one of these points is fixed, which we may assume to be $<u>$. But then $F_{<u,e>} \leqslant F_{<u^*>} = F_{<e_i>} \cong \hat{B}_4(\mathbb{F})$.

We now proceed to the "twisted" cases. π may be viewed as the set of points of its "algebraic closure", $\bar{\pi} = \pi \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ ($\bar{\mathbb{F}}$ an algebraic closure of $\mathbb{F}$) fixed under the Frobenius map $Frob_q$ acting as $\alpha \mapsto \alpha^q$ on the coefficients of elements with respect to a given base, say e_1,e_2,e_3. By a well-known consequence (cf. SPRINGER [11]) of Lang's result on the vanishing of the first order Galois cohomology, the group $\bar{F}_\sigma = C_{\bar{F}}(\sigma)$ of all elements of $\bar{F}$ (the algebraic closure of F) which are fixed by a given automorphism σ of the form $\tau \circ Frob_q$ for some inner automorphism τ of F, is isomorphic to F. Similarly, the $\bar{F}_\sigma$-module $\bar{V}_\sigma$ is isomorphic to the F-module V. We shall apply this principle to find orbits in twisted versions $\bar{\pi}_\sigma$ of π.

(B.6) (*Triality Twisted Version*). Let τ be the member of $T \cong \mathrm{Sym}_3$ which induces (123) on $\{e_1,e_2,e_3\}$. Then $\bar{\pi}$ is normalized by $\sigma = \tau \circ Frob_q$, so $\bar{\pi}_\sigma$ is well defined. It has $\mathbb{F}$-basis $f_1 = e$, $f_2 = \lambda e_1 + \lambda^q e_2 + \lambda^{-1-q} e_3$, $f_3 = \lambda^q e_1 + \lambda^{-1-q} e_2 + \lambda e_3$, where $\lambda \in \bar{\mathbb{F}}$ is a fixed primitive $(q^2 + q + 1)$-st root of unity. All points $<u>$ of $\bar{\pi}_\sigma$ satisfy $\mathcal{Q}(u) \neq 0$. This follows from the fact that no point in $P(\overline{< e_i,e_j >})$ is fixed by σ. Thus, any line contained in a triality twisted special plane contains only black points. It is readily seen that, for $<u> \in P(\bar{\pi}_\sigma)$, we have

$$^3D_4(\mathbb{F}) \cong C_{\bar{F}_\sigma}(\bar{\pi}) \leqslant (\bar{F}_\sigma)_{<u>} \leqslant (\bar{F}_\sigma)_{<e,u>} \leqslant N_{\bar{F}_\sigma}(\bar{\pi}_\sigma) \cong {}^3D_4(\mathbb{F}).3$$

(and, by the argument above, we may identify $\bar{F}_\sigma$ with F and $\bar{\pi}_\sigma$ with a plane of $\mathbb{K}$ containing $< e >$).

As in the untwisted case, we need to know, for τ in $N_{\bar{F}_\sigma}(\bar{\pi}_\sigma)$ inducing (123) on $\{e_1,e_2,e_3\}$, and $u \in \bar{\pi}_\sigma$, when

(*) $<e,\tau(u)> \equiv <e,u> \mod <e>$.

A simple computation, using $\tau(f_2) = (\lambda^q + \lambda^{-1-q} + \lambda)e - f_2 - f_3$ and $\tau(f_3) = f_2$ shows that (*) is satisfied with $u = \alpha f_2 + \beta f_3$ ($\alpha,\beta \in \mathbb{F}$) if and only if $\alpha^2 - \alpha\beta + \beta^2 = 0$. We consider the separate cases of q modulo 3.

(i). $q \equiv 1 \pmod 3$. The above equation in α,β has two distinct solutions up to scalar multiples. These lead to two orbits with $F_{<e,u>} \cong {}^3D_4(\mathbb{F}).3$. The remaining $(q-1)/3$ orbits have $F_{<e,u>} \cong {}^3D_4(\mathbb{F})$.

(ii). $q \equiv 2 \pmod 3$. There are no solutions, so we get $(q+1)/3$ orbits with $F_{<e,u>} \cong {}^3D_4(\mathbb{F})$.

(iii). $q \equiv 0 \pmod 3$. The solution $\alpha = -\beta$ leads to a single orbit with stabilizer $F_{<e,u>} \cong {}^3D_4(\mathbb{F}).3$. The remaining $q/3$ orbits have $F_{<e,u>} \cong {}^3D_4(\mathbb{F})$.

(B.7) (*The Duality Twist*). We now take σ to be the composition of the Frobenius automorphism $Frob_q$ with an automorphism $\tau \in T$ which induces (23) on $\{e_1,e_2,e_3\}$. As in (B.4), $\bar{\pi}$ is σ-invariant. The fixed point space $\bar{\pi}_\sigma$ has $\mathbb{F}$-basis $f_1 = e$, $f_2 = e_2 + e_3$, $f_3 = \lambda e_2 + \lambda^{-1} e_3$, where $\lambda \in \bar{\mathbb{F}}$ is a primitive $(q+1)$-st root of unity. It has a unique point $<u>$ for which $u^\# = 0$, namely $<u> = < e_1 >$ and $\mathcal{G}(\bar{\pi}_\sigma) = P(< f_2,f_3 >)$. For $u \in \bar{\pi}_\sigma \setminus (< e_1 > \cup < f_2,f_3 >)$, we have

$$^2\hat{D}_4(\mathbb{F}) \cong (\bar{F}_\sigma) \leqslant (\bar{F}_\sigma)_{<u>} \leqslant (\bar{F}_\sigma)_{<e,u>} \leqslant N_{\bar{F}_\sigma}(\bar{\pi}_\sigma) \cong {}^2\hat{D}_4(\mathbb{F}).2.$$

Setting $u = \alpha f_2 + f_3$ ($\alpha \in \mathbb{F}$) we get that $<e,\tau(u)> \equiv <e,u> \mod <e>$ if and only if

(+) $2\alpha = -(\lambda + \lambda^{-1})$.

There are two cases to consider.

(i). $q \equiv 0 \pmod 2$. Then (+) has no solutions, so there are $q/2$ orbits with stabilizer $F_{<e,u>} \cong {}^2D_4(\mathbb{F})$.
(ii). $q \equiv 1 \pmod 2$. There is a unique solution to (+), leading to a simple orbit with stabilizer $F_{<e,u>} \cong {}^2\hat{D}_4(\mathbb{F}).2$. The remaining $(q-1)/2$ orbits have $F_{<e,u>} \cong {}^2\hat{D}_4(\mathbb{F})$.

Since $u^{\#} = (\alpha + \lambda)(\alpha + \lambda^{-1})e_1$, we must have $F_{<e,u>} \leqslant F_{<e_1>} \cong \hat{B}_4(\mathbb{F})$ for all $<u>$ considered. This ends the proof of (B.1).

We end this section with the following consequence of (B.1).

(B.8). *The $\tilde{E}$-orbits of lines containing black but no white points are as indicated in Table 3, with the exception that, if $q = 2$, the members of $\mathcal{L}_9$ lie in a special plane but have no black points.*

Table 3.
Lines l black but no white points.

orbit	F-type of l	stabilizer $\hat{E}_l$	$\tilde{E}$-orbit size	$dist(l)$
$\mathcal{L}_7$	III	$[q^{15}].G_2(\mathbb{F}).\mathbb{F}^*$	$\dfrac{q^{15}(q^{12}-1)(q^9-1)(q^8-1)(q^5-1)}{q-1}$	$(0,1,q)$
$\mathcal{L}_8$	IV	$[q^7].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$	$\dfrac{q^{20}(q^{12}-1)(q^9-1)(q^4+1)(q^5-1)}{q-1}$	$(0,2,q-1)$
$\mathcal{L}_9$	V-VIII	$\hat{D}_4(\mathbb{F}).\mathrm{Sym}_3$	$\dfrac{q^{24}(q^{12}-1)(q^9-1)(q^8-1)(q^5-1)}{6(q^4-1)^2}$	$(0,3,q-2)$
$\mathcal{L}_{10}$	IX-X	${}^3D_4(\mathbb{F}).3$	$\frac{1}{3}q^{24}(q^9-1)(q^8-1)(q^5-1)(q^4-1)$	$(0,0,q+1)$
$\mathcal{L}_{11}$	XI-XII	${}^2\hat{D}_4(\mathbb{F}).2$	$\frac{1}{2}q^{24}(q^{12}-1)(q^9-1)(q^5-1)$	$(0,1,q)$

Proof. Let $l \in P_2$ be a line with $\mathcal{W}(l) = \varnothing \neq \mathcal{B}(l)$. By transitivity of $\tilde{E}$ on $\mathcal{B}$, we may assume $<e> \in l$, and so, according to (B.1), l has type I, II,..., or XII. As for types I and II, the lines $l = <e,x>$ in (B.2), (B.3) belong, respectively, to $\mathcal{L}_3$ and $\mathcal{L}_4$. The distributions $dist(l)$, stabilizers E_l, and actions of E_l on $P(l)$ were completely determined in section W.
Suppose $l = <e,x>$ has type III, and $<x> \in \mathcal{G}(l)$. Then $\hat{E}_l$ is transitive on l because if $<f> \in \mathcal{B}(l) \setminus <e>$, then $\mathcal{D}(f)$ is a cube in $\mathbb{F}^*$, so there is $g \in \hat{E}$ with $f^g = e$ and, by transitivity of F on the set of lines labeled III, we may take g such that $<x>^g = <x>$. Thus $F_l = F_{<e,x>}$ has index q in $\hat{E}_l$.
Next, suppose that l has type IV. Then, up to conjugacy, we may take $\mathcal{G}(l) = \{<x>,<y>\}$, where $x = e_2 + e_3, y = e_1 + e_3^1$. Since $(x^{\#},y) \neq 0$ and $(x,y^{\#}) = 0$, we have $E_l = E_{<x>,<y>}$. Using appropriate elements of H, we find that E_l has $(3,q-1)$ orbits of equal length on $\mathcal{B}(l)$, and that $E_l = F_{<e,x>}.\mathbb{F}^*$.
It remains to discuss the case where l lies in one of the (twisted or untwisted) special planes π of Section B. Using the fact that the number of point orbits in π equals the number of line orbits in π (or just the automorphisms d_α and $h \in H$ from Section A), we get that $\tilde{E}_\pi$ acts transitively on each of the sets $\pi \cap \mathcal{W}(l)$, $\pi \cap \mathcal{G}(l)$, $\pi \cap \mathcal{B}(l)$, and on the set of all lines $m \subset \pi$ with $\mathcal{W}(m) = \varnothing$ and $\mathcal{B}(m) \neq \varnothing$. Therefore, l belongs to one of the $\tilde{E}$-orbits listed and the stabilizer in $\tilde{E}$ is as stated.

The orbits found are named $\mathcal{L}_7, \ldots, \mathcal{L}_{11}$ as indicated in the table.

Section G. GRAY LINES.

We complete our determination of $\tilde{E}$-orbits of lines in this section. Since we have already found all orbits of lines which contain white or black points, it remains to determine the $\tilde{E}$-orbits on $\mathcal{G}_2$.

(G.1). *The group E is transitive on $\mathcal{L}_{12} = \{l \in \mathcal{G}_2 : l_{\mathcal{S}} \neq \emptyset\}$. Moreover, if $l \in \mathcal{G}_2(S)$, then $\hat{E}_l = \hat{E}_{S,l} \cong [q^{16}].{}^2\hat{D}_4(\mathbb{F}).\mathbb{Z}_{q+1}.2$.*

Proof. If $l \in \mathcal{G}_2$ and $l_{\mathcal{S}} \neq \emptyset$, then $|l_{\mathcal{S}}| = 1$. Now E is transitive on $\mathcal{S}$, and if S is a symplecton, then we have $\hat{E}_S / R_u(\hat{E}_S) \cong \hat{D}_5(\mathbb{F}).\mathbb{F}^*$ is transitive on $\mathcal{G}_2(S)$ (cf. (P.2)). The second assertion follows from standard knowledge of the D_5-space S.

(G.2). *Let $S_1, S_2 \in \mathcal{S}$ with $S_1 \cap S_2 = <x> \in \mathcal{W}$. Then $\hat{E}_{S_1,S_2} \cong [q^{16}].\hat{D}_4(\mathbb{F}).(\mathbb{F}^*)^2$ is transitive on each of the following three sets of pairs: $\{y_1 \in S_1, y_2 \in S_2 : x \times y_1 = x \times y_2 = 0\}$; $\{y_1 \in S_1, y_2 \in S_2 : x \times y_1 = 0, x \times y_2 \neq 0\}$; and $\{y_1 \in S_1, y_2 \in S_2 : x \times y_1 \neq 0, x \times y_2 \neq 0\}$. The stabilizer in $\hat{E}_{S_1,S_2}$ of a such a pair is isomorphic to $[q^{14}].G_2(\mathbb{F}).(\mathbb{F}^*)^2$, $[q^8].\hat{B}_3(\mathbb{F}).\mathbb{F}^*$, $\hat{D}_4(\mathbb{F})$, in the respective cases.*

Proof. The unipotent radical $R_u(\hat{E}_{S_1,S_2})$ fixes each line m in S_i on x with $x \times m = 0$ (and, for each line m, is transitive on $m \setminus \{x\}$). The quotient $\hat{E}_{S_1,S_2} / R_u(\hat{E}_{S_1,S_2}) \cong \hat{D}_4(\mathbb{F}).(\mathbb{F}^*)^2$ acts on the two 8-spaces $<z \in S_i : z \times x = 0> / <x>$, $i = 1,2$ with inequivalent actions. (This can be seen directly by the fact that $\hat{E}_{S_1,S_2}$ preserves the map from $\mathcal{W}_2(S_1 \cap x') \cap x_{\mathcal{W}_2}$ onto $\mathcal{W}_5(S_2 \cap x') \cap x_{\mathcal{W}_5}$ given by $l \mapsto <S_2 \cap l'>$; this map induces an incidence preserving map from singular points of the D_4-space $<z \in S_1 : z \times x = 0> / <x>$ onto one of the classes of singular 4-spaces of the D_4-space $<z \in S_2 : z \times x = 0> / <x>$). The group E_{S_1,S_2} has two orbits on the set of points $y_2 \in \mathcal{G}(S_2)$ which are distinguished by $y_2 \times x = 0$ and $y_2 \times x \neq 0$, respectively. Moreover, the stabilizer in $\hat{E}_{S_1,S_2}$ of $y_2 \in S_2$ is $\hat{E}_{S_1,y_2} \cong [q^{15}].\hat{B}_3(\mathbb{F}).(\mathbb{F}^*)^2$ if $x \times y_2 = 0$, and $\hat{E}_{S_1,y_2} \cong [q^8].\hat{D}_4(\mathbb{F}).\mathbb{F}^*$, otherwise. (G.2) readily follows from knowledge of the action of $\hat{E}_{S_1,y_2} \leqslant \hat{E}_{S_1,S_2}$ on S_1.

(G.3). *Let $\mathcal{L}_{13}$ be the collection of lines $<y_1,y_2> \in \mathcal{G}_2$ which are not contained in a special plane and satisfy $<y_1>_{\mathcal{S}} \cap <y_2>_{\mathcal{S}} \in \mathcal{W}$. Then E is transitive on $\mathcal{L}_{13}$ with stabilizer of $l \in \mathcal{L}_{13}$ in $\hat{E}$ isomorphic to $[q^{14}].(G_2(\mathbb{F}) \times SL_2(\mathbb{F})).\mathbb{F}^*$. If $q > 2$, this is the single orbit of lines $<y_1,y_2> \in \mathcal{G}_2$ with $<y_1>_{\mathcal{S}} \cap <y_2>_{\mathcal{S}} \in \mathcal{W}$; if $q = 2$, there is exactly one additional such orbit, namely $\mathcal{L}_9$.*

Proof. For $l = <y_1,y_2> \in \mathcal{L}_{13}$, set $S_i = <y_i>_{\mathcal{S}}$, $i = 1,2$. Thus $S_1 \cap S_2 = <x> \in \mathcal{W}$. In view of (G.2), there are only three cases to be considered (up to interchanging y_1 and y_2).

(i). $x \times y_1 = x \times y_2 = 0$. We may take $y_1 = e|_2 + e|_1$ and $y_2 = e|_3 + e|_1$. A direct check (using elements of the form $s_{g_1,g_2,1}$) shows that E_l induces $PSL_2(\mathbb{F})$ on $P(l)$. From this we conclude that $\hat{E}_l \cong [q^{14}].(G_2(\mathbb{F}) \times SL_2(\mathbb{F})).\mathbb{F}^*$.

(ii). $x \times y_1 = 0$, $x \times y_2 \neq 0$. We may take $y_1 = e|_2 + e|_1$ and $y_2 = e_2 + e_3$. Now $<y_1 + y_2> \in \mathcal{B}(l)$, a contradiction to the hypothesis.

(iii). $x \times y_1 \neq 0$, $x \times y_2 \neq 0$. The triple $<x,y_1,y_2>$ spans a special plane. Therefore, if $z \in P(l)$ satisfies $z \notin P(<x,x_1>) \cup P(<x,x_2>) \cup P(<x_1,x_2>)$, then $z \in \mathcal{B}$. Such points exist if and only if $q > 2$. This ends the proof of (G.3).

Remark. The stabilizer E_l of $l \in \mathcal{L}_{13}$ is contained in the stabilizer of a point, namely E_x for $x = \bigcap_{y \in l} y_{\mathcal{S}}$.

It now remains to consider lines l in $\mathcal{G}_2$ for which there are points $y_1, y_2 \in P(l)$ satisfying $<y_1>_{\mathcal{S}} \cap <y_2>_{\mathcal{S}} \in \mathcal{W}_5$.

(G.4). *Let $l \in \mathcal{G}_2$, $<y_1>$, $<y_2> \in \mathcal{G}(l)$. Write $S_i = \{y_i\}_{\mathcal{S}}$ and $U = S_1 \cap S_2$. If $U \in \mathcal{W}_5$ and there is a hyperplane A of U with $y_i \times A = 0$ for each $i=1,2$, then $\mathcal{W}(l) \neq \varnothing$.*

Proof. We have $\dim\{y_i \in S_i:\ A \times y_i = 0\} = 6$ for both $i = 1,2$. Since $U = (S_1 \cap A') \cap (S_2 \cap A')$, we have that $C = <S_1 \cap A', S_2 \cap A'>$ has dimension 7. However, if $B = A_{\mathcal{W}_6}$, then $B_i = S_i \cap B \in \mathcal{W}_5$ $(i=1,2)$ satisfy $<B_1,B_2> = B$ and $B_1 \cap B_2 = A$. Therefore, B is a hyperplane in C, and, as $<y_1,y_2> \subseteq C$, that $B \cap <y_1,y_2> \neq 0$. Hence $\mathcal{W}(<y_1,y_2>) \neq \varnothing$.

(G.5). *Let $x,x_1,x_2 \in \mathcal{W}$ be such that $x \in \Gamma(x_1) \cap \Gamma(x_2)$ and set $S_i = \{x,x_i\}_{\mathcal{S}}$. Assume $S_1 \cap S_2 \in \mathcal{W}_5$. If $x_1 \in <x_2>'$ and $<y_i> \in \mathcal{G}(<x,x_i>)$ for each $i = 1,2$, then $\mathcal{W}(<y_1,y_2>) \neq \varnothing$.*

Proof. Observe that $<y_i>_{\mathcal{S}} = S_i$ and $U = S_1 \cap S_2 \in \mathcal{W}_5$. Since $x_1 \in <x_2>'$, we get from (P.6) $<x_1>' \cap S_2 = <U \cap <x_1>', x_2> = <U \cap <x_2>', x_1>$, whence $U \cap <x_1>' = U \cap <x_2>' = A$. Then $y_i \times A = 0$, and so, by (G.4), $\mathcal{W}(<y_1,y_2>) \neq \varnothing$.

(G.6). *The group E is transitive on the set of triples x,x_1,x_2 of distinct white points with $x \in \Gamma(x_1) \cap \Gamma(x_2)$, $x_1 \in \Gamma(x_2)$, and $S_1 \cap S_2 \in \mathcal{W}_5$ where $S_i = \{x,x_i\}_{\mathcal{S}}$. The stabilizer in $\hat{E}$ of such a triple is isomorphic to $[q^{20}].SL_3(\mathbb{F}).(\mathbb{F}^*)^3$ and is transitive on the set of pairs $y_1 \in \mathcal{G}(<x,x_1>)$, $y_2 \in \mathcal{G}(<x,x_2>)$.*

Proof. Set $S = S(x_1,x_2)$, $U = S_1 \cap S_2$, and $U_i = S \cap S_i$, for $i = 1,2$. Since $U \cap <x_i>'$ is a hyperplane in U contained in $S \cap S_1 \cap S_2$, we have $S \cap S_1 \cap S_2 \in \mathcal{W}_3$, and so, by (P.5), $U_i \in \mathcal{W}_5$. Clearly, $x \in U$ and $x_i \in U_i$ for $i = 1,2$. Now, E is transitive on $\mathcal{W}_3$ and if $N \in \mathcal{W}_3$, then $E_N \cong [q^{29}].(SL_2(\mathbb{F}) \times SL_3(\mathbb{F}) \times SL_3(\mathbb{F}))$. $(\mathbb{F}^*)^3$ is transitive on triples U, U_1, $U_2 \in N_{\mathcal{MW}_5}$ such that $U \not\subseteq <U_1,U_2>$, with stabilizer in $\hat{E}$ isomorphic to $[q^{29}].(SL_2(\mathbb{F}) \times SL_3(\mathbb{F})$. $(\mathbb{F}^*)^3$ in $\hat{E}$.
Put $N = <e_1, e^1_{12}, e^1_{13}>$, $U = <N, e^2_{12}, e^3_{13}>$, $U_1 = <N, e^3_{22}, e^3_{23}>$, and $U_2 = <N, e^3_{32}, e^3_{33}>$. Using this explicit description and automorphisms s_{g_1,g_2,g_3} and $t_{x,y}$ as described in Section A, it is readily seen that E_{N,S,S_1,S_2} is transitive on the set of all triples of points $x \in P(U \setminus N)$, $x_1 \in P(U_1 \setminus x')$, $x_2 \in P(U_2 \setminus (x' \cup <x_1>'))$. Now $x = <e^3_{12}>$, $x_1 = <e^3_{23}>$, $x_2 = <e^3_{32} + e^3_{33}>$ is such a triple and their stabilizer is $\hat{E}_{x,x_1,x_2} \cong [q^{20}].SL_3(\mathbb{F}).(\mathbb{F}^*)^3$. These three points are the only white points in $<x,x_1,x_2>$ and all other points are gray. Finally, elements from H show that the stabilizer of x,x_1,x_2 is transitive on pairs $y_1 \in \mathcal{G}(<x,x_1>)$, $y_2 \in \mathcal{G}(<x,x_2>)$. This completes the result (G.6).

(G.7). *E is transitive on the set $\mathcal{L}_{14}$ of lines $l \in \mathcal{G}_2$ such that, if $y_1,y_2 \in l$ and $y_1 \neq y_2$, then $(y_1)_{\mathcal{S}} \cap (y_2)_{\mathcal{S}} \in \mathcal{W}_5$. Moreover, $[E_l,E_l]$ induces $PSL_2(\mathbb{F})$ on $P(l)$ and $\hat{E}_l \cong [q^{23}].(SL_2(\mathbb{F}) \times SL_3(\mathbb{F}))$. $\mathbb{F}^*$.*

Proof. Let $U = S_1 \cap S_2 \in \mathcal{W}_5$. By (G.5), if A_1,A_2 are hyperplanes of U such that $y_i \times A_i = 0$, then $A_1 \neq A_2$. Let $x \in P(U \setminus (A_1 \cup A_2))$. Then $y_i \times x \neq 0$ for each $i=1,2$. By (W.1) and (W.5), $|\mathcal{W}(<x,y_i>)| = 2$. Let $\{x_i\} = \mathcal{W}(<x,y_i>) \setminus \{x\}$, $i = 1,2$. Now x,x_1,x_2 satisfies the hypotheses of (G.6). We deduce that $\hat{E}$ is transitive on $\mathcal{L}_{14}$, and, for l a representative, E_l induces $PSL_2(\mathbb{F})$ on l.

We summarize

(G.8). *Every purely gray line belongs to one of the three $\hat{E}$-orbits in Table 4.*

Table 4.
$\hat{E}$-orbits of purely gray lines

orbit type	stabilizer $\hat{E}_l$	orbit size	example
$\mathscr{L}_{12}$	$[q^{16}].{}^2\hat{D}_4(q).(q^2-1).2$	$\frac{q^8(q^{12}-1)(q^9-1)(q^5-1)}{2(q^2-1)}$	off quadric in a symplecton
$\mathscr{L}_{13}$	$[q^{14}].(G_2(\mathbb{F})\times SL_2(\mathbb{F})).\mathbb{F}^*$	$\frac{q^{15}(q^{12}-1)(q^9-1)(q^8-1)(q^5-1)}{(q^2-1)(q-1)}$	$< e^1_{21}+e^1_{12}, e^1_{13}+e^1_{31} >$
$\mathscr{L}_{14}$	$[q^{23}].(SL_2(\mathbb{F})\times SL_3(\mathbb{F})).\mathbb{F}^*$	$\frac{q^9(q^{12}-1)(q^9-1)(q^8-1)(q^5-1)(q^3+1)}{(q^2-1)(q-1)}$	$< e^1_{12}+e^1_{23}, e^1_{12}+e^1_{32}+e^1_{33} >$

This completes the results.

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